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On eddy viscosity models that restrict the dynamics to the larger eddies

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Abstract. The very essence of large eddy simulation (LES) is that the LES-solution contains only scales of size $\geq \delta$, where δ is a user-chosen length scale. Therefore, in case the LES is based on an eddy viscosity model we determine the eddy viscosity such that any scales of size $< \delta$ are dynamically insignificant. In this paper, we address the following two questions: how much eddy diffusion is needed to (a) (counter)balance the production of scales of size smaller than δ ; and (b) damp any disturbances having a scale of size smaller than δ initially. From this we deduce that the eddy viscosity ν_e has to depend on the invariants $q = \frac{1}{2}\text{tr}(S^2)$ and $r = -\frac{1}{3}\text{tr}(S^3)$ of the strain rate tensor S . The simplest model is then given by $\nu_e = \frac{3}{2}(\delta/\pi)^2|r|/q$. This model is successfully tested for a turbulent channel flow ($\text{Re}_\tau=590$).

1. Introduction

1.1. Large-eddy simulation

Large-eddy simulation (LES) seeks to predict the dynamics of spatially filtered turbulent flows. Therefore a spatial filter is applied to the (incompressible) Navier-Stokes (NS) equations,

$$\partial_t \bar{u} + (\bar{u} \cdot \nabla) \bar{u} + \nabla \bar{p} - 2\nu \nabla \cdot S(\bar{u}) = \nabla \cdot (\bar{u} \bar{u}^T - \overline{uu^T}), \quad (1)$$

where it is assumed that the filter $u \mapsto \bar{u}$ commutes with differentiation. The right-hand side represents the effects of the residual scales on the ‘large eddies’ (the part of the fluid motion with velocity \bar{u}). It depends on both u and \bar{u} , due to the nonlinearity. The dependence on u is removed by introducing a closure model $\tau(\bar{u}) \approx \overline{uu^T} - \bar{u} \bar{u}^T$. The motion of the larger eddies is then governed by

$$\partial_t v + (v \cdot \nabla) v + \nabla \bar{p} - 2\nu \nabla \cdot S(v) = -\nabla \cdot \tau(v) \quad (2)$$

Here the variable name is changed from \bar{u} to v to stress that the solution of Eq. (2) differs from that of Eq. (1), because the closure model is not exact. The inequality $\tau(v) \neq \overline{uu^T} - \bar{u} \bar{u}^T$ is crucial, since information is to be lost: the solution v of Eq. (2) must possess less scales of motion (degrees of freedom) than the Navier-Stokes solution u , see also Guermond et al. (2004). Finding a closure model that is both inexact (to reduce the complexity of the flow) and accurate (to approximate the dynamics of the larger eddies) represents the main difficulty to LES. Since turbulence is so far from being completely understood, there is a wide range of closure models, mostly based on heuristic, *ad hoc* arguments that cannot be derived from the NS-equations, see

for example Sagaut (2001) and the references therein. The most commonly used closure model is given by

$$\tau(v) = -2\nu_e S(v) \quad (3)$$

where ν_e denotes the eddy viscosity. The classical Smagorinsky model reads $\nu_e = C_S^2 \delta^2 \sqrt{4q}$ where $q(v) = \frac{1}{2}\text{tr}(S^2(v))$ is an invariant of the strain rate tensor $S(v)$. Here it may be remarked that ν_e is commonly expressed in terms of $|S(v)| = \sqrt{2\text{tr}(S(v)^2)} = \sqrt{4q}$. Further, it may be noticed that the Smagorinsky model depends only on the length δ of the filter, and not on the details of the map $u \mapsto \bar{u}$. Various value for the Smagorinsky constant C_S have been proposed, mainly ranging from $C_S = 0.1$ to $C_S = 0.17$ Pope (2000). Instead of adhering to a constant value one can also take $C_S = C_S(v)$. In the well-known dynamical procedure, for instance, the coefficient C_S is computed with the help of the Jacobi identity (in least-square sense).

1.2. Problem setting

The solution v of Eq. (2) is composed of eddies of different size. The very essence of large eddy simulation is that v contains only eddies of size $\geq \delta$, where δ is the user-chosen length of the filter $u \mapsto \bar{u}$. This property enables us to solve (2) numerically when it is not feasible to compute the Navier-Stokes solution, i.e., the full turbulent flow field u . Therefore we view the closure model τ as a function of v that is to be determined such that the dynamically significant scales of motion in the solution v of Eq. (2) are greater than (or equal to) δ . That is, the closure model τ is designed to eliminate all scales of length $< \delta$. In the present approach we try not to make any specific assumptions (about spectra, e.g.). Rather, we address the question: “when does the closure model stop the production of smaller scales of motion from continuing at the filter scale?” In this way we find Condition (7), see below, which ensures that the transfer of energy *from* the large eddies to the subfilter scales is balanced properly. This condition is necessary, but not sufficient, to limit the dynamics governed by Eq. (2) to scales of size $\geq \delta$. To that end, the energy that is transferred from the subfilter scales *to* the large eddies, should be dynamically insignificant too. Therefore, we superimpose subfilter-scale perturbations to a velocity field v that does not contain subfilter scales initially, and require that the perturbation dissipates at a natural rate. This yields Condition (8), below. The two conditions given by Eqs. (7)-(8) guarantee that the large scales of motion are separated from the smaller scales. The basic idea is to determine the eddy viscosity in such a manner that these two conditions are satisfied.

2. When does the closure model stop the production of smaller scales of motion from continuing at the filter scale?

We consider an arbitrary part Ω_δ with diameter δ of the flow domain and take the filtered velocity \bar{u} equal to the average of u over Ω_δ . Furthermore, we suppose that Ω_δ is a periodic box, so that boundary terms resulting from integration by parts (in the computations to come) vanish. Poincaré’s inequality states that there exists a constant C_δ , depending only on Ω_δ , such that for every function v in the Sobolev space $W^{1,2}(\Omega_\delta)$,

$$\int_{\Omega_\delta} ||v - \bar{v}||^2 dx \leq C_\delta \int_{\Omega_\delta} ||\nabla v||^2 dx \quad (4)$$

The optimal constant C_δ - the Poincaré constant for the domain Ω_δ - is the inverse of the smallest (non-zero) eigenvalue of the dissipative operator $-\nabla^2$ on Ω_δ , see for example Courant & Hilbert (1989). Payne & Weinberger (1960) have shown that the Poincaré constant is given by $C_\delta = (\delta/\pi)^2$ for convex domains Ω_δ .

The residual field $v' = v - \bar{v}$ contains eddies of size smaller than δ . These small scales are produced by the nonlinear, convective term in Eq. (2). The closure model must keep them from

becoming dynamically significant. Poincaré's inequality (4) shows that the $L^2(\Omega_\delta)$ norm of the residual field v' is bounded by a constant (independent of v) times the $L^2(\Omega_\delta)$ norm of ∇v . Consequently, we can confine the dynamically significant part of the motion to scales $\geq \delta$ by controlling the velocity gradient. To see how the evolution of the $L^2(\Omega_\delta)$ norm of ∇v is to be restrained by the closure model, we consider the residual field v' first:

$$\frac{d}{dt} \int_{\Omega_\delta} \frac{1}{2} \|v'\|^2 dx = \int_{\Omega_\delta} (-\nu \|\nabla v'\|^2 + T(\bar{v}, v') + \tau'(v) : S(v')) dx$$

Here, the second term in the right-hand side stands for the energy transfer from \bar{v} to v' ; the third term represents the eddy dissipation, i.e., the dissipation resulting from the closure model. Eq. (2) should not produce subfilter scales, i.e., the eddy dissipation has to balance the energy transfer at the scale set by the filter. Now suppose that the closure model is taken such that the last two terms in the right-hand side above cancel each other out. Then we have

$$\frac{d}{dt} \int_{\Omega_\delta} \frac{1}{2} \|v'\|^2 dx = -\nu \int_{\Omega_\delta} \|\nabla v'\|^2 dx \quad (5)$$

This equation shows that the evolution of the energy of v' is not depending on \bar{v} . Stated otherwise, the energy of subfilter scales dissipates at a natural rate, without any forcing mechanism involving scales larger than δ . In this way, the scales $< \delta$ are separated from scales $\geq \delta$. With the help of the Poincaré inequality (4) and the Gronwall lemma, we obtain from Eq. (5) that

$$\int_{\Omega_\delta} \frac{1}{2} \|v'\|^2(x, t) dx \leq \exp(-2\nu t/C_\delta) \int_{\Omega_\delta} \frac{1}{2} \|v'\|^2(x, 0) dx$$

In other words, the energy of the subfilter scales decays at least as fast as $\exp(-2\nu t/C_\delta)$, for *any* filter length δ . Applying Poincaré's inequality and Gronwall's lemma to

$$\frac{d}{dt} \int_{\Omega_\delta} \frac{1}{2} \|\nabla v\|^2 dx \leq -\nu \int_{\Omega_\delta} \|\nabla^2 v\|^2 dx \quad (6)$$

results into the same rate of decay. So, in conclusion, we can keep the subfilter component v' under control by imposing (6). Integration by parts of the L^2 product of Eq. (2) with $\nabla^2 v$ yields

$$\frac{d}{dt} \int_{\Omega_\delta} \frac{1}{2} \|\nabla v\|^2 dx = -\nu \int_{\Omega_\delta} \|\nabla^2 v\|^2 dx + \int_{\Omega_\delta} (v \cdot \nabla) v \cdot \nabla^2 v dx - \int_{\Omega_\delta} \tau(v) : S(\nabla^2 v) dx$$

Thus we see that Eq. (6) holds if the last term dominates the convective contribution. Chae (2005) showed that

$$\int_{\Omega_\delta} (v \cdot \nabla) v \cdot \nabla^2 v dx = 4 \int_{\Omega_\delta} r(v) dx$$

where $r(v) = -\frac{1}{3} \text{tr}(S^3(v)) = -\det S(v)$ is an invariant of the strain rate tensor $S(v)$. Hence, the dissipative condition (6) holds if

$$\boxed{\int_{\Omega_\delta} \tau(v) : S(v) dx \leq -4C_\delta \int_{\Omega_\delta} r(v) dx} \quad (7)$$

Here it may be remarked again that C_δ is the inverse of the smallest (non-zero) eigenvalue of $-\nabla^2$ on Ω_δ . In conclusion, the dissipation introduced by the closure model ($\tau(v) : S(v)$) counteracts the nonlinear production of scales $< \delta$ if the closure is taken according to Eq. (7). Notice that the production can be quantified with the help of the invariant $r(v)$ of $S(v)$.

3. When does the closure model damp subfilter-scale disturbances properly?

If $r(v) \leq 0$ we can take $\tau = 0$ according to Eq. (7). This is because Condition (7) only ensures that the transfer of energy *from* the large eddies to the subfilter scales is balanced properly by the eddy dissipation. To see if effect of subfilter scales *on* the large eddies is modelled properly too, we suppose that the velocity field v does not contain subfilter scales initially, i.e., at time $t = 0$ we have $v = \bar{v}$ in an arbitrary part Ω_δ with diameter δ of the flow domain. Now, we superimpose an instantaneous, solenoidal, subfilter-scale perturbation δv to v on Ω_δ . Initially one may conceive the unperturbed field $v = \bar{v}$ as being constant on Ω_δ , whereas the perturbation δv is any (non-constant) periodic function on Ω_δ . The evolution of the perturbed velocity $v + \delta v$ is governed by Eq. (2) with v replaced by $v + \delta v$. Consequently, the evolution of the $L^2(\Omega_\delta)$ norm of δv is given by

$$\frac{d}{dt} \int_{\Omega_\delta} \frac{1}{2} \|\delta v\|^2 dx = \int_{\Omega_\delta} (-\nu \|\nabla \delta v\|^2 - \delta v \cdot S(v) \delta v + [\tau(v + \delta v) - \tau(v)] : S(\delta v)) dx$$

It may be emphasised that this expression is exact, that is we need not assume that δv is small. The middle term in the right-hand side represents the energy transfer from v to δv . As before the closure is taken such that it neutralises/dominates this term,

$$\boxed{\int_{\Omega_\delta} [\tau(v + \delta v) - \tau(v)] : S(\delta v) dx \leq \int_{\Omega_\delta} \delta v \cdot S(v) \delta v dx} \quad (8)$$

Then, we have

$$\frac{d}{dt} \int_{\Omega_\delta} \frac{1}{2} \|\delta v\|^2 dx \leq -\nu \int_{\Omega_\delta} \|\nabla \delta v\|^2 dx$$

This equation shows that the evolution of the energy of the perturbation δv is not depending on the unperturbed, base flow v . Compare Eq. (5). As a result, the energy of subfilter scales dissipates at a natural rate (without any nonlinear mechanism involving scales larger than δ). Once again with the help of the Poincaré inequality (4) and Gronwalls lemma, we obtain that the energy of subfilter disturbances decays at least as fast as $\exp(-2\nu t/C_\delta)$, for *any* filter length δ . So, in conclusion, the LES-model given by Eq. (2) is stable with respect to subfilter disturbances - i.e., the backward transfer of energy is properly closed - if Eq. (8) holds.

4. A unified condition for the eddy dissipation

Condition (7) can also be expressed in terms of the vorticity $\omega = \nabla \times v$. By taking the curl of Eq. (2) we find the vorticity equation and from that we obtain that the enstrophy is governed by

$$\frac{d}{dt} \int_{\Omega_\delta} \frac{1}{2} \|\omega\|^2 dx = -\nu \int_{\Omega_\delta} \|\nabla \omega\|^2 dx + \int_{\Omega_\delta} \omega \cdot S\omega dx + \int_{\Omega_\delta} \nabla \times \tau(v) : S(\omega) dx$$

In the right-hand side we recognise the vortex stretching term that can produce smaller scales of motion and the eddy dissipation that should counteract the production of smaller scales at the scale δ . Since $v = \bar{v} + v'$ with \bar{v} constant in Ω_δ (by definition), we can make use of the approximation $\nabla \times \tau(v) = \nabla \times (\tau(\bar{v}) + \frac{d\tau}{dv}(\bar{v}) v' + \dots) \approx \frac{d\tau}{dv}(\bar{v}) \omega$ in Ω_δ . Thus, with the help of the identity $\int_{\Omega_\delta} \|\nabla v\|^2 dx = \int_{\Omega_\delta} \|\omega\|^2 dx$ we see that Eq. (7) can (in lowest order) be written as

$$\int_{\Omega_\delta} \frac{d\tau}{dv}(\bar{v}) \omega : S(\omega) dx \leq - \int_{\Omega_\delta} \omega \cdot S(v) \omega dx \quad (9)$$

The essential difference between this inequality and Eq. (8) is the sign of the transport term: if we would simply disregard the difference between δv with ω and name them both ϕ , then Eq. (8) and Eq. (9) become (in lowest order)

$$\boxed{\int_{\Omega_\delta} \frac{d\tau}{dv}(\bar{v}) \phi : S(\phi) dx \leq \pm \int_{\Omega_\delta} \phi \cdot S(v) \phi dx} \quad (10)$$

where the plus sign corresponds to Eq. (8) and the minus sign to Eq. (9). From a physical point of view the minus sign represents the requirement that the forward cascade of energy stops at the scale δ set by the filter (that is, the production of smaller scales stops at the filter scale); the plus sign expresses that there is no backward cascade too (that is, scales of size $< \delta$ cannot become dynamically relevant, since any subfilter perturbations decay exponentially fast). If the minus sign is taken ϕ has to be a solution of the vorticity equation, else ϕ is governed by the perturbation equation. In summary, we get (in lowest order)

$$\partial_t \phi + (v \cdot \nabla) \phi \pm (\phi \cdot \nabla) v - 2\nu \nabla \cdot S(\phi) + \nabla \psi = -\nabla \cdot \frac{d\tau}{dv}(\bar{v}) \phi$$

where the gradient term is to be omitted in case of the minus sign; otherwise the incompressibility constraint $\nabla \cdot \phi = 0$ is to be imposed. Consequently, the $L^2(\Omega_\delta)$ norm of ϕ is governed by

$$\frac{d}{dt} \int_{\Omega_\delta} \frac{1}{2} \|\phi\|^2 dx + \nu \int_{\Omega_\delta} \|\nabla \phi\|^2 dx = \pm \int_{\Omega_\delta} \phi \cdot S(v) \phi dx + \int_{\Omega_\delta} \frac{d\tau}{dv}(\bar{v}) \phi : S(\phi) dx$$

The left-hand side is required to be negative (for both $\phi = \omega$ and $\phi = \delta v$), see before; hence the right-hand side has to be negative too. This yields Eq. (10).

5. Towards an eddy viscosity model

For the eddy viscosity model given by Eq. (3) condition (10) becomes

$$-\nu_e \int_{\Omega_\delta} \|\nabla \phi\|^2 dx \leq \pm \int_{\Omega_\delta} \phi \cdot S(v) \phi dx \quad (11)$$

Furthermore, $C_\delta \int_{\Omega_\delta} \|\nabla \phi\|^2 dx \geq \int_{\Omega_\delta} \|\phi\|^2 dx$, where the equality holds if ϕ is aligned with the eigenfunction associated with the smallest eigenvalue of $-\nabla^2$ on Ω_δ . Thus we obtain $\nu_e \int_{\Omega_\delta} \|\phi\|^2 dx \geq \pm C_\delta \int_{\Omega_\delta} \phi \cdot S(v) \phi dx$. So, in conclusion, *the eddy viscosity ν_e depends on the Poincaré constant C_δ and the two non-zero invariants $q(v) = \frac{1}{2} \text{tr}(S^2(v))$ and $r(v) = -\det(S(v))$ of the strain rate tensor $S(v)$* . To see how ν_e depends on $q(v)$ and $r(v)$, we consider the minus sign in the right-hand side of Eq. (11). This corresponds to the choice $\phi = \omega$. In that case, it can be deduced that $\int_{\Omega_\delta} \phi \cdot S(v) \phi dx = 4 \int_{\Omega_\delta} r(v) dx$, see Chae (2005) for details. Here it may be remarked that the calculations by Chae are done for the 3D Euler equations; yet one can add the viscous term to each step of Chae's calculations. Hence Eq. (11) with $\phi = \omega$ (i.e., with the minus sign in rhs) is equivalent to

$$\nu_e \int_{\Omega_\delta} q(\omega) dx \geq \int_{\Omega_\delta} r(v) dx \quad (12)$$

The left-hand side can be bounded in terms of v ,

$$\int_{\Omega_\delta} q(\omega) dx = \int_{\Omega_\delta} \frac{1}{4} \|\nabla \omega\|^2 dx \geq \frac{1}{C_\delta} \int_{\Omega_\delta} \frac{1}{4} \|\omega\|^2 dx = \frac{1}{C_\delta} \int_{\Omega_\delta} q(v) dx$$

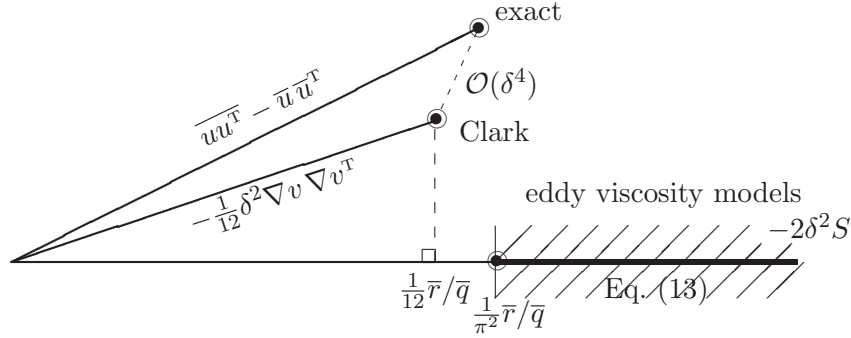


Figure 1. Some LES-models in the space of symmetric 3x3 tensors.

where the equality sign holds if ω is fully aligned with the eigenfunction of the dissipative operator $-\nabla^2$ on Ω_δ associated with the smallest non-zero eigenvalue. Consequently, the eddy viscous term in Eq. (12) dominates the nonlinear, convective term if

$$\boxed{\nu_e \int_{\Omega_\delta} q(v) dx \geq C_\delta \int_{\Omega_\delta} r(v) dx} \quad (13)$$

It may be emphasised here that Condition (13) can also be derived directly from Eqs. (3)-(7).

It has not been established, thus far, that the choice of the minimal eddy viscosity satisfying Eq. (13), i.e.,

$$\nu_e = C_\delta \overline{r(v)/q(v)}, \quad (14)$$

will adequately model the subfilter contributions to the evolution of the filtered velocity. With the help of the Navier-Stokes solution u we can analyze the consistency of the approximation $\overline{u u^T} - \overline{u u^T} \approx 2\nu_e S(\overline{u})$. A series expansion gives $\overline{u u^T} - \overline{u u^T} = -\frac{\delta^2}{12}(\nabla \overline{u} \nabla \overline{u}^T) + \mathcal{O}(\delta^4)$. The leading term is known as the Clark model, see Clark et al. (1979). Unfortunately, the Clark model cannot be used as a stand-alone LES model, since it produces a finite time blow-up of the kinetic energy (Vreman et al. (1996)). Projecting both Eq. (14) and the Clark model onto $S(v)$ leads to the following consistency question

$$-2C_\delta \frac{\bar{r}}{\bar{q}} \int_{\Omega_\delta} S(v) : S(v) dx \stackrel{?}{=} -\frac{\delta^2}{12} \int_{\Omega_\delta} \nabla v \nabla v^T : S(v) dx. \quad (15)$$

The integral in the right-hand side equals $4 \int_{\Omega_\delta} r(v) dx$ (Chae (2005)). This shows that r provides a measure of the alignment of the Clark model and S . By definition we have $S : S = 2q$. Consequently, Eq. (15) shows that the order of the modeling error is optimal if $C_\delta = \delta^2/12$. This value is in fair agreement with the Poincaré constant, $C_\delta = \delta^2/\pi^2$; yet, it is slightly lower. The overall situation is sketched in Figure 1. The horizontal axis in this figure represents all possible eddy viscosity models; the axis is parameterized by the eddy viscosity. The shaded part of the horizontal axis in Figure 1 depicts the subset of eddy viscosities that satisfy Eq. (13).

It may be stressed that, thus far, we have assumed that $\overline{r(v)}$ is positive; notice that $q(v)$ is non-negative by definition. The projection of the Clark model on S is positive if and only if $\overline{r(v)} > 0$. Stated otherwise the Clark model yields anti-dissipation if $\overline{r(v)} < 0$, implying that it is unstable with respect to perturbations δv having a scale of size δ initially. Eq. (13) has been derived from Eq. (11) where we have taken the minus sign in the right-hand side. It restricts the eddy viscosity only if $\int_{\Omega_\delta} \phi \cdot S(v) \phi dx = 4 \int_{\Omega_\delta} r(v) dx > 0$. In case this expression is negative, we have to consider the plus sign in the right-hand side of Eq. (11). Since the eddy viscosity has to

depend on C_δ , $q(v)$ and $r(v)$, we base the eddy viscosity ν_e on the absolute value of $r(v)$. That is, we take

$$\boxed{\nu_e \int_{\Omega_\delta} q(v) dx = C_\delta \left| \int_{\Omega_\delta} r(v) dx \right|} \quad (16)$$

Eq. (16) yields $\nu_e = 0$ in any (part of the) flow where $r = 0$. That is, the eddy viscosity vanishes if the nonlinear transport to/from scales $< \delta$ is absent. At a no-slip wall $r = 0$ too; hence $\nu_e = 0$ at the wall. In homogeneous, isotropic turbulence, we have $r/q \propto Re^{1/2}$. Therefore $\nu_e/\nu \propto Re^{3/2}$ for fixed δ . Additionally, we obtain that $\nu_e + \nu \rightarrow \nu$ if $\nu \propto Re^{-1} \propto \delta^2 r/q \propto \delta^2 Re^{1/2}$, that is if $\delta \propto Re^{-3/4}$. This shows that the eddy viscosity given by Eq. (16) vanishes as δ is of the order of $Re^{-3/4}$, i.e., if δ approaches the Kolmogorov scale.

6. A simple qr-model

To compute the eddy viscosity ν_e according to Eq. (16), we need know how q and r vary within Ω_δ . Here, we cannot simply take $\overline{q(v)} = q(\bar{v})$, because the relation between q and v is nonlinear. On the other hand, however, we do not want to compute v' explicitly. This problem is similar to the closure problem in LES, except that the original closure problem concerns the residual of the Navier-Stokes solution u , whereas here it is about the residual of the large-eddy solution v . It may be noted that we can also compute the eddy viscosity directly from Eq. (16) provided that the grid is taken such that the residual v' is fully resolved numerically. Obviously, this implies that the grid size is to be taken smaller than the filter width δ . Therefore the computational costs will be higher than usual. This approach is successfully tested for decaying isotropic turbulence (the Comte-Bellot & Corrsin experiment at $Re_\lambda = 71.6$), see Verstappen et al. (2011) for more details. Here we apply an approximate deconvolution method that recovers some of the information lost in the filtering process, see Berselli et al. (2006), e.g. To recover an approximation for v' we consider the series expansion of v around \bar{v} . Ignoring terms that are of the order δ^4 , we get the approximation $v' \approx -\frac{1}{24} \delta^2 \nabla^2 \bar{v}$. Notice that the deconvolution method is commonly applied to approximate the subfilter part of the Navier-Stokes solution u , whereas it is used to approximate the subfilter part of the LES-velocity v here. In homogeneous, isotropic turbulence we have $r \propto Re^{3/2}$ and $q \propto Re^{1/2}$; hence the ratio of r and $q^{3/2}$ scales like Re^0 . This scaling law suggests to take $\overline{r(v)/q(v)}^{3/2} \approx r(\bar{v})/q(\bar{v})^{3/2}$. Thus Eq. (16) leads to

$$\nu_e \approx C_\delta \frac{|r(\bar{v})|}{q(\bar{v})^{3/2}} \left(\frac{\overline{q(v)}}{q(\bar{v})} \right)^{1/2} \sqrt{q(\bar{v})}$$

Furthermore with the help of the approximate deconvolution method and the Poincaré inequality (4) it can be shown that

$$\begin{aligned} \overline{q(v)} &= \frac{1}{4} \|\nabla v\|^2 \approx \frac{1}{4} \|\nabla(\bar{v} - \frac{1}{24} \delta^2 \nabla^2 \bar{v})\|^2 \\ &\stackrel{(4)}{\leq} \frac{1}{4} (1 + \frac{1}{24} \delta^2 / C_\delta)^2 \|\nabla \bar{v}\|^2 = c^2 \overline{q(\bar{v})} \end{aligned}$$

with $c = 1 + \pi^2/24 \approx \frac{3}{2}$, where the equality-sign holds (once again) if v is fully aligned with eigenfunction of $-\nabla^2$ on Ω_δ associated with the eigenvalue $1/C_\delta$. Since $\overline{q(v)} \approx q(\bar{v}) = q(v) + \mathcal{O}(\delta^2)$ we obtain (in lowest order) the eddy viscosity model

$$\boxed{\nu_e(v) = \frac{3}{2} C_\delta \frac{|r(v)|}{q(v)}} \quad (17)$$

Eq. (17) is invariant under rotation of coordinate axis, since it depends on the invariants of $S(v)$. The eddy viscosity model (17) can be put into the standard notation by introducing the relation

$$C_S^2(v) = \frac{3}{4\pi^2} \frac{|r(v)|}{\sqrt{q^3(v)}} \quad (18)$$

In homogeneous, isotropic turbulence we have $C_S^2 \propto r/\sqrt{q^3} \propto Re^0$, i.e., the Smagorinsky coefficient is (in lowest order) independent of the Reynolds number Re . So, if we average Eq. (18) over the homogeneous directions we obtain an approximately constant coefficient C_S^2 that is valid for a wide range of Reynolds numbers (in case of homogeneous, isotropic turbulence). This partially agrees with Smagorinsky's reasoning, in which C_S^2 is taken constant (once again: provided that $r \propto Re^{3/2}$ and $q \propto Re^{1/2}$).

The eigenvalues of the symmetric tensor S are real-valued. Therefore the invariants are constrained by $27r^2 - 4q^3 \leq 0$. Consequently, Eq. (17) yields an eddy viscosity in the range

$$0 \leq \nu_e \leq \frac{1}{2\pi^2\sqrt{3}} \delta^2 |S|$$

This shows that the largest value of the Smagorinsky coefficient C_S is equal to $1/\sqrt{2\pi^2\sqrt{3}} \approx 0.17$. Remarkably this maximum value is identical to Lilly's value, $C_S = 0.17$, see Lilly (1967), which implies that the standard Smagorinsky model with $C_S = 0.17$ has (more than) sufficient eddy dissipation. This upper bound was also found by means of other reasoning, see Meneveau & Lund (1997). Interestingly, the value $C_S = 0.17$ has been found too large in many numerical experiments. In turbulent shear flow, for instance, the value of the coefficient C_S is often reduced to the relatively low value $C_S = 0.1$ to give the standard model a fair change for success.

7. First results

In summary, the eddy viscosity model given by Eq. (17) has the following properties: (a) $\nu_e = 0$ in any (part of the) flow where $r = 0$, i.e., the eddy viscosity vanishes if the nonlinear transport to/from scales $< \delta$ is absent; hence (b) $\nu_e = 0$ in any 2D flow; (c) $\nu_e = 0$ at a wall; (d) $\nu_e \rightarrow 0$ if $\delta \propto Re^{-3/4}$; (e) $C_S \leq 0.17$. It goes without saying that the performance of the eddy viscosity model (17) has to be investigated for many cases. As a first step it was tested for turbulent channel flow by means of a comparison with direct numerical simulations. This flow forms a prototype for near-wall turbulence: virtually every LES has been tested for it. The results are compared to the DNS data of Moser et al. (1999) at $Re_\tau = 590$. In fact, we should compare the LES-solution v to the filtered DNS-solution \bar{u} . Yet, since the filtered DNS-solution is not given by Moser et al. (1999) we will compare v directly to u . The dimensions of the channel are taken identical to those of the DNS of Moser *et al.* The computational grid used for the large-eddy simulation consists of 64^3 points. The DNS was performed on a $384 \times 257 \times 384$ grid, i.e., the DNS uses about 144 times more grid points than the present LES. The LES-results were obtained with an incompressible code that uses a fourth-order, symmetry-preserving, finite-volume discretization, see Verstappen & Veldman (2003).

The eddy viscosity model given by Eq. (17) has been derived for continuous variables. A discrete representation of Eq. (17) may be derived along similar lines. To that end both the PDE's (2)-(3) and Conditions (7)-(8) are to be discretized. The resulting discrete conditions may then be worked out in the manner of the continuous condition, but now in the discrete setting, yielding a discrete representation of Eq. (17). Obviously, the result will depend on the details of the discretization method that is applied to Eqs. (2)-(3). Generically, it will again be of the form given by Eq. (17) with q and r replaced by the invariants of the discrete rate-of-strain tensor, and the Poincaré constant C_δ replaced by the inverse of the eigenvalue, corresponding to

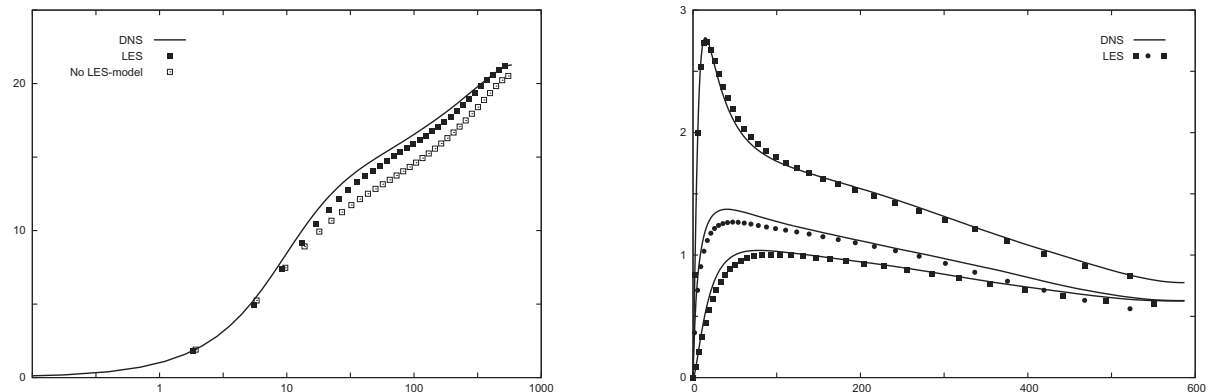


Figure 2. The left-hand figure shows the mean velocity (in wall coordinates) obtained with the help of the 64^3 LES and the DNS by Moser et al. (1999). Results obtained on the 64^3 LES-grid without closure model (i.e., $\nu_e = 0$) are also shown for reference (open symbols). The right-hand figure displays the root-mean-square of the fluctuating velocities. The boxes and circles represent LES data; every symbol corresponds to data in a grid point.

the scale δ , of the discrete approximation of $-\nabla^2$. To explain our approximation of the Poincaré constant, we consider a second-order central discretization on a uniform grid with spacing dx , dy and dz . The largest eigenvalue of the discrete approximation of $-\nabla^2$ is then given by

$$\mu_{max} = \frac{4}{dx^2} + \frac{4}{dy^2} + \frac{4}{dz^2}$$

This eigenvalue describes the greatest possible damping in the numerical simulation, i.e., it provides a measure for the dissipation at the scale of the grid cell. Hence if we take Ω_δ equal to the grid cell and approximate the smallest eigenvalue of $-\nabla^2$ on Ω_δ by μ_{max} , we arrive at the following approximate relation:

$$C_\delta \approx \frac{1}{\mu_{max}} \quad (19)$$

Thus in case $dx = dy = dz = h$, we get $\delta \approx h$. In case the grid is nonuniform, μ_{max} can be approximate locally by multiplying the discrete dissipative operator with the mode associated with the highest frequency that fits on the grid (i.e., the +1,-1, +1 mode). With the help of Eq. (19) we can compute δ for a given grid (and discretization of the dissipative operator $-\nabla^2$). It may be noted that the resulting relation between δ and the grid width differs from the usual expression $\delta = (dx dy dz)^{1/3}$ if the grid is (strongly) nonuniform. The eddy viscosity model (17) is essentially not more complicated to implement in a LES-code than the standard Smagorinsky model (with C_S constant). Indeed, the model (17) is expressed in terms of the invariants of the strain rate tensor and does not involve explicit filtering. The invariant $q = \frac{1}{4}|S|^2$ is to be computed in any case; the computation of r is just as difficult. Unlike the standard Smagorinsky model (even with the relatively low value $C_S = 0.1$), the present model showed an appropriate behavior. As can be seen in Fig. 2 both the mean velocity and the root-mean-square of the fluctuating velocity are in good agreement with the DNS. To illustrate how much the eddy viscosity model contributes to the quality of the solution, the mean velocity profile obtained on the 64^3 LES-grid without closure model (i.e., $\nu_e = 0$) is also shown in Fig. 2.

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